SOME GENERALIZATIONS OF C_2 (C_3) MODULES

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Abstract. The notion of extending modules was generalized to purely extending modules by John Clark in 1998. An R-module M is said to be *purely extending (or* $\mathrm{P.C}_1$*)* if every submodule of M_R is essential in a pure submodule. In this paper, we introduce pure analogues of the C_2 and C_3 properties. Specifically, we define a module M_R to be *purely* C_2 *(denoted by* $\mathrm{P.C}_2$*)* (resp. *generalized purely* C_2*, denoted by* $\mathrm{G.P.C}_2$*)* if every submodule isomorphic to a pure submodule (resp. direct summand) of M_R is itself pure. Furthermore, we say that M_R satisfies the *purely* C_3 (resp. *generalized purely* C_3, denoted by $\mathrm{G.P.C}_3$) property if the sum of any two independent pure submodules (resp. direct summands) of M_R is a pure submodule. In addition, we establish that $\mathrm{G.P.C}_2 \Rightarrow \mathrm{G.P.C}_3$, whereas the converse does not hold in general. The following diagram illustrates these implications and highlights the irreversibility of certain relationships:

$$
\begin{tikzcd}
 & \mathrm{C}_2 \arrow[r, shift right, "/" marking] \arrow[d, shift right] & \mathrm{C}_3 \arrow[l, shift right, "/" marking] \arrow[d, shift right] \arrow[dr, shift left, "/" marking] & \\
\mathrm{P.C}_2 \arrow[ur, "/" marking] \arrow[r] & \mathrm{G.P.C}_2 \arrow[u, shift right, "/" marking] \arrow[r, shift right] & \mathrm{G.P.C}_3 \arrow[l, shift right, "/" marking] \arrow[u, shift right, "/" marking] \arrow[r, shift left, "/" marking] & \mathrm{P.C}_3 \arrow[l, shift left] \arrow[ul, shift left, "/" marking]
\end{tikzcd}
$$

Focusing on modules satisfying the $\mathrm{P.C}_1$ condition, we investigate their general properties and the interrelations among the following classes:

- M_R is *purely continuous* (or *p.continuous*) if it satisfies both $\mathrm{P.C}_1$ and $\mathrm{P.C}_2$.
- M_R is *purely quasi-continuous* (or *p.quasi-continuous*) if it satisfies both $\mathrm{P.C}_1$ and $\mathrm{P.C}_3$.
- M_R is *generalized purely continuous* (or *g.p.continuous*) if it satisfies both $\mathrm{P.C}_1$ and $\mathrm{G.P.C}_2$.
- M_R is *generalized purely quasi-continuous* (or *g.p.quasi-continuous*) if it satisfies both $\mathrm{P.C}_1$ and $\mathrm{G.P.C}_3$.

Furthermore, we demonstrate that every quasi-continuous module is p.quasi-continuous, and every quasi-injective module is p.continuous. A comprehensive study of these modules is provided, along with several established properties.

1. MAIN RESULTS

In this section, we investigate the general properties of modules satisfying the $\mathrm{P.C}_i$ and $\mathrm{G.P.C}_i$ conditions for $2 \leq i \leq 3$, alongside their interrelations. We provide several examples of such modules. To facilitate our study, we first recall that a submodule N of M_R is said to be *closed* if it admits no proper essential extensions in M_R; equivalently, N is maximal with respect to the property that $N \cap K = 0$ for some $K \leq M_R$. Throughout this paper, we denote that N is a closed submodule of M_R by $N \leq_c M_R$. The notations $N \subseteq M$, $N \leq M_R$, $N \leq_{ess} M_R$, $N \leq_{\oplus} M_R$ and $N \leq_p M_R$ mean that N is a subset, a submodule, an essential submodule, a direct

2020 *Mathematics Subject Classification.* 16D50, 16E50, 16D99.

Key words and phrases. Closed submodule, continuous module, direct summand, g.p.continuous module, g.p.quasi-continuous module, pure submodule, p.continuous module, p.quasi-continuous module, quasi-continuous module, von Neumann regular ring.

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summand and a pure submodule of M_R, respectively.
For the reader's convenience and to provide a clear connection between the characterizations of C_1 and $\mathrm{P.C}_1$ modules, we recall the following results from [4] and [1].

Proposition 1.1. [4, Proposition 6.32, Lemma 6.41] *The following statements are equivalent:*

(a) *M_R has C_1 property;*
(b) *every closed submodule of M_R is a direct summand;*
(c) *if $A \leq_{\oplus} \mathrm{E}(M)$, then $A \cap M \leq_{\oplus} M_R$.*

Proposition 1.2. [1, Lemma 1.1] *The following statements are equivalent:*

(a) *M_R has $\mathrm{P.C}_1$ property;*
(b) *every closed submodule of M_R is a pure submodule;*
(c) *if $A \leq_{\oplus} \mathrm{E}(M)$, then $A \cap M \leq_p M_R$.*

The preceding propositions imply that $\mathrm{C}_1 \Rightarrow \mathrm{P.C}_1$, however, the converse does not hold in general, as shown by the following example. To this end, we first state the following necessary result.

Proposition 1.3. *Let R be a commutative PID and M be an R-module.*

(a) *$\mathrm{T}(M) \leq_p M_R$.*
(b) *If M_R is torsion free, then M_R has $\mathrm{P.C}_1$ property.*

The next example shows that the torsion free condition in Proposition 1.3(b) is necessary.

Example 1.4. There exists a torsion abelian group G which has not $\mathrm{P.C}_1$ property. For example, let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_8$. We shall show that $N =< (\overline{1}, \overline{2}) >$ is a closed submodule of $M_{\mathbb{Z}}$, however it is not a pure submodule because $4N = 0 \neq 4M \cap N$. We claim that $N \leq_c M_{\mathbb{Z}}$. To see this, first we note that N is not essential in $M_{\mathbb{Z}}$. In addition, it is routine to check that $C :=< (\overline{1}, 0), (0, \overline{2}) >$ is the only proper submodule of $M_{\mathbb{Z}}$ such that $N \subsetneq C$. While N can not be essential in $C_{\mathbb{Z}}$ because $N \leq_{\oplus} C_{\mathbb{Z}}$. Therefore N is a closed submodule of $M_{\mathbb{Z}}$. Therefore by Proposition 1.2, $G_{\mathbb{Z}}$ is not $\mathrm{P.C}_1$.

Proposition 1.5. *Let R be a ring. The following statements are equivalent:*

(a) *R is a von Neumann regular ring;*
(b) *every (cyclic) right ideal of R is a pure submodule of R_R;*
(c) *every short exact sequence of R-modules is pure;*
(d) *every (cyclic) R-module is flat.*

As previously noted, we now provide an example demonstrating that $\mathrm{P.C}_1 \not\Rightarrow \mathrm{C}_1$.

Example 1.6. (a) By Proposition 1.3, every torsion free abelian group has $\mathrm{P.C}_1$ property, while it has not C_1 property in general. For instance, the abelian group $M = \mathbb{Z}^{\aleph_\circ}$ is $\mathrm{P.C}_1$ which is not C_1 (for more details, see [4, Example 4.88]).
(b) There exists a commutative von Neumann regular continuous ring R such that $\mathrm{M}_2(R)$ is neither left nor right extending, see [1, p. 355]. While $\mathrm{M}_2(R)$ is a von Neumann regular ring and so by Proposition 1.5, it is right and left purely extending.

In the following Theorem, we establish that the $\mathrm{G.P.C}_2$ property implies $\mathrm{G.P.C}_3$; however, we shall also provide an example to demonstrate that the converse does not hold in general.

Theorem 1.7. *Let M be an R-module. If M_R has* G.P.C_2 *property, then it has* G.P.C_3 *property. In particular, every g.p.continuous module is g.p.quasi-continuous.*

Before proceeding, we mention the following proposition, which establishes that for finitely generated modules over Noetherian rings, the notions of P.C_i (G.P.C_i) and C_i for $2 \leq i \leq 3$, coincide.

Proposition 1.8. *Let R be a right Noetherian ring and M be a finitely generated R-module. The following statements hold:*

(a) *every pure submodule of M_R is a direct summand;*
(b) *M_R has* P.C_1 *property if and only if M_R has* C_1 *property;*
(c) *M_R satisfies* P.C_i (G.P.C_i) $(2 \leq i \leq 3)$ *if and only if M_R satisfies* C_i.

We note that the G.P.C_2 and G.P.C_3 properties do not hold in general. To illustrate this, we provide the following example.

Example 1.9. (a) The abelian group $\mathbb{Z}$ does not have C_2 (P.C_2, G.P.C_2) property. Because $\mathbb{Z} \simeq 2\mathbb{Z}$ however $2\mathbb{Z}$ is not a pure submodule of $\mathbb{Z}_{\mathbb{Z}}$.
(b) Let $M = \mathbb{Z} \oplus \mathbb{Z}$. We show that $M_{\mathbb{Z}}$ does not have C_3 (P.C_3, G.P.C_3) property. To see this, let $A = \mathbb{Z}$, and $B =< (1, 2) >$. Clearly A and B are direct summands of $M_{\mathbb{Z}}$. While $A \oplus B = \mathbb{Z} \oplus 2\mathbb{Z}$ is not a pure submodule of $M_{\mathbb{Z}}$ because $2\mathbb{Z}$ is not pure in $\mathbb{Z}_{\mathbb{Z}}$.

We refer the reader to the diagram provided in the Abstract, which elucidates the relationships between these concepts. To avoid redundancy, we omit the reproduction of the diagram here; however, we shall provide several examples demonstrating that the implications presented therein are not reversible.

Example 1.10. (a) Let R be a von Neumann regular ring which is not right injective. By Proposition 1.5, every R-module has all of the properties which are written in the second row of the diagram. While $R \oplus \mathrm{E}(R)$ has not even C_3 property because R is not a direct summand of $\mathrm{E}(R)$. Therefore any condition in the second row of the above diagram does not imply the conditions in the first row.
(b) There exists an R-module M with C_3 (and so G.P.C_3) property while it has not P.C_3 property. Therefore $C_3 \not\Rightarrow$ P.C_3 and G.P.$C_3 \not\Rightarrow$ P.C_3. For instance, there exists a countable torsion free indecomposable abelian group G with $\mathrm{rank}(G) = n > 1$. For more details, see [2, Last remark] and [3, Section 3]. We recall that $\mathrm{rank}(G)$ means the cardinal number of a maximal independent system containing only elements of infinite and prime power orders. Since $G_{\mathbb{Z}}$ is indecomposable, it has C_3 property. We claim that $G_{\mathbb{Z}}$ has not P.C_3 property. To see this, on the contrary, assume that $G_{\mathbb{Z}}$ has P.C_3 property. By Proposition 1.3, $G_{\mathbb{Z}}$ has P.C_1 property. Therefore $G_{\mathbb{Z}}$ is p.quasi-continuous and so it is quasi-continuous(over right Noetherian rings, the notions of p.quasi-continuous and quasi-continuous coincide). Thus by [5, p. 19], $G_{\mathbb{Z}}$ should be isomorphic to a subgroup of $\mathbb{Q}$. This contradicts with $\mathrm{rank}(G) > 1$.

(c) This example is showing that G.P.$C_3 \not\Rightarrow$ G.P.C_2. Let R be any commutative domain which is not a field. Therefore R_R is quasi-continuous and so it has C_3 property. Thus R_R has G.P.C_3 property. While by Proposition 1.11, R_R has not G.P.C_2.

(d) Let X be any non trivial subgroup of $\mathbb{Q}$. It is clear that $X_{\mathbb{Z}}$ is quasi-continuous and so by Theorem 1.14, it has P.C_3 property. It is routine to prove that $X_{\mathbb{Z}}$ has not P.C_2 property. Therefore P.$C_3 \not\Rightarrow$ P.C_2.

We next demonstrate that for a domain R, the properties of being continuous, p.continuous, and g.p.continuous are identical for R_R.

Theorem 1.11. *Let R be a domain. The following statements are equivalent:*

(a) *R_R is continuous;*
(b) *R_R is p.continuous;*
(c) *R_R is g.p.continuous;*
(d) *R_R has C_2 property;*
(e) *R_R has $\mathrm{P.C}_2$ property;*
(f) *R_R has $\mathrm{G.P.C}_2$ property;*
(g) *R is a division ring.*

In the following, we characterize p.quasi-continuous modules. To facilitate a comparison with the existing characterization of quasi-continuous modules, we first recall the known results for the latter.

Theorem 1.12. [5, Theorem 2.8] *The following are equivalent for an R-module M:*

(a) *M_R is quasi-continuous;*
(b) *for any two submodules X and Y which are complements of each other, $M = X \oplus Y$;*
(c) *$\mathrm{E}(M) = \oplus_{i\in I} E_i$ implies that $M = \oplus_{i\in I}(E_i \cap M)$.*

Theorem 1.13. *The following are equivalent for an R-module M:*

(a) *M_R is p.quasi-continuous;*
(b) *for any two submodules X and Y which are complements of each other, $X \oplus Y \leq_p M_R$;*
(c) *$\mathrm{E}(M) = \oplus_{i\in I} E_i$ implies that $\oplus_{i\in I}(E_i \cap M) \leq_p M_R$.*

As noted in Example 1.10(b), the C_3 property does not necessarily imply $\mathrm{P.C}_3$. However, in the following corollary, we show that the implication $\mathrm{C}_3 \Rightarrow \mathrm{P.C}_3$ is recovered under the C_1 condition. Consequently, every quasi-continuous module is p.quasi-continuous.

Theorem 1.14. (a) *Every quasi-continuous module is p.quasi-continuous.*
(b) *Every quasi-injective module is p.continuous.*

The relationships between these notions are summarized in the following diagram:

$$\begin{tikzcd}
\text{Quasi-injective} \arrow[r] \arrow[d] & \text{Continuous} \arrow[r] \arrow[d] & \text{Quasi-continuous} \arrow[d] \arrow[dr] & \\
\text{p.continuous} \arrow[r] & \text{g.p.continuous} \arrow[r] & \text{g.p.quasi-continuous} & \text{p.quasi-continuous} \arrow[l]
\end{tikzcd}$$

We observe that $\mathrm{C}_2 \Rightarrow \mathrm{C}_3$. A natural question arises as to whether $\mathrm{P.C}_2 \Rightarrow \mathrm{P.C}_3$. To address this question, we conclude our discussion with the following proposition.

Proposition 1.15. *Let M be an R-module with $\mathrm{P.C}_2$ property. For any two submodules M_1 and M_2 of M_R such that $M_1 \cap M_2 = 0$, if $M_1 \leq_\oplus M_R$ and $M_2 \leq_p M_R$, then $M_1 \oplus M_2 \leq_p M_R$.*

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